The H can be chosen* in such a manner that $R(\rho)$ will have one of the following forms

$$\text{I.}\quad R(\rho) = f(\rho^2)$$

$$\text{II.}\quad R(\rho) = f(\rho^2)\sqrt{\rho^2 - 1};\; R(\rho) = f(\rho^2)\sqrt{\rho^2 - \eta^2};\; R(\rho) = f(\rho^2)\rho;$$

$$\text{III.}\quad R(\rho) = f(\rho^2)\sqrt{\rho^2 - \eta^2};\; R(\rho) = f(\rho^2)\sqrt{\rho^2 - 1};$$
$$R(\rho) = f(\rho^2)\sqrt{\rho^2 - 1}\sqrt{\rho^2 - \eta^2};$$

$$\text{IV.}\quad R(\rho) = f(\rho^2)\rho\sqrt{\rho^2 - 1}\sqrt{\rho^2 - \eta^2},$$

with analogous expressions for $M(\mu)$ and $N(\nu)$. In this case the product $V = RMN$ will be a harmonic polynomial satisfying the equation of Laplace. It must be remarked that there exist $2n + 1$ functions of Lamé of degree n.

Thus the functions of first order are 3, namely:

$$R_1 = \sqrt{\rho^2 - 1},$$
$$R_2 = \sqrt{\rho^2 - \eta^2},$$
$$R_3 = \rho$$

and analogous functions for M and N.

There are 5 functions of the second order:

$$R_4 = \rho\sqrt{\rho^2 - \eta^2}$$
$$R_5 = \rho\sqrt{\rho^2 - 1}$$
$$R_6 = \sqrt{\rho^2 - 1}\sqrt{\rho^2 - \eta^2}$$

(analogous expressions for M and N), and also two functions R_7 and R_8 (M_7 and M_8, N_7 and N_8) which must have the form I, i.e., must be polynomials.

In order to avoid making use of the characteristic equation, we shall find the functions by the consideration that their product must be a harmonic polynomial.

Let the roots of the polynomials be α_1 and α_2; then, substituting in the fundamental equality (4) α_i for λ^2 $(i = 1,2)$, we obtain

* H must be in this event root of a certain algebraic equation known as the characteristic equation.

$$(\rho^2 - \alpha_i)(\mu^2 - \alpha_i)(\nu^2 - \alpha_i) = C\left(\frac{x^2}{\alpha_i - 1} + \frac{y^2}{\alpha_i - \eta^2} + \frac{z^2}{\alpha_i} - 1\right)$$

$$C = (\alpha_i - 1)(\alpha_i - \eta^2)\alpha_i.$$

Since the polynomial in x^2, y^2, z^2 on the right side must be harmonic, we obtain, using the Laplacian of the right side of the equation,

$$\frac{1}{\alpha_i - 1} + \frac{1}{\alpha_i - \eta^2} + \frac{1}{\alpha_i} = 0. \tag{9}$$

$\alpha_i (i = 1,2)$ will be the root of the equation

$$3\alpha^2 - 2(1 + \eta^2)\alpha + \eta^2 = 0. \tag{10}$$

We get

$$\alpha_1 = \frac{1 + \eta^2 + \sqrt{1 - \eta^2 + \eta^4}}{3}$$
$$\alpha_2 = \frac{1 + \eta^2 - \sqrt{1 - \eta^2 + \eta^4}}{3} \tag{11}$$

α_1 and α_2 lie within the limits

$$1 > \alpha_1 > \eta^2 > \alpha_2 \geq 0.$$

Limiting the expansion to the fourth power or less of

$$\alpha_1 = \frac{2}{3} + \frac{1}{6}\eta^2 + \frac{1}{8}\eta^4 + \cdots$$
$$\alpha_2 = \frac{1}{2}\eta^2 - \frac{1}{8}\eta^4 + \cdots. \tag{12}$$

The polynomials of degree zero are equal to 1:

$$R_0 = M_0 = N_0 = 1$$

Now we may compile a table containing the eight functions of Lamé. The letter n denotes here the degree of the function, and k the order of the function in accordance with the designation of Poincaré (see Table 2).

TABLE 2. THE FIRST EIGHT FUNCTIONS OF LAMÉ

n	k	R_k	M_k	N_k	$R_k\ M_k\ N_k$	Remark
0	0	1	1	1	1	
1	1	$\sqrt{\rho^2-1}$	$\sqrt{1-\mu^2}$	$\sqrt{1-\nu^2}$	C_1x	$C_1 = \sqrt{1-\eta^2}$
1	2	$\sqrt{\rho^2-\eta^2}$	$\sqrt{\mu^2-\eta^2}$	$\sqrt{\eta^2-\nu^2}$	C_2y	$C_2 = \eta\sqrt{1-\eta^2}$
1	3	ρ	μ	ν	C_3z	$C_3 = \eta$
2	4	$\rho\sqrt{\rho^2-\eta^2}$	$\mu\sqrt{\mu^2-\eta^2}$	$\nu\sqrt{\eta^2-\nu^2}$	C_4yz	$C_4 = C_2C_3$
2	5	$\rho\sqrt{\rho^2-1}$	$\mu\sqrt{1-\mu^2}$	$\nu\sqrt{1-\nu^2}$	C_5zx	$C_5 = C_3C_1$
2	6	$\sqrt{\rho^2-\eta^2}\cdot\sqrt{\rho^2-1}$	$\sqrt{\mu^2-\eta^2}\sqrt{1-\mu^2}$	$\sqrt{\eta^2-\nu^2}\sqrt{1-\nu^2}$	C_6xy	$C_6 = C_1C_2$
2	7	$\rho^2-\alpha_1$	$\mu^2-\alpha_1$	$\nu^2-\alpha_1$	$C_7\left(\frac{x^2}{\alpha_1-1} + \frac{y^2}{\alpha_1-\eta^2} + \frac{z^2}{\alpha_1} - 1\right)$	$C_7 = \alpha_1(\alpha_1-\eta^2)(\alpha_1-1)$
2	8	$\rho^2-\alpha_2$	$\mu^2-\alpha_2$	$\nu^2-\alpha_2$	$C_8\left(\frac{x^2}{\alpha_2-1} + \frac{y^2}{\alpha_2-\eta^2} + \frac{z^2}{\alpha_2} - 1\right)$	$C_8 = \alpha_2(\alpha_2-\eta^2)(\alpha_2-1)$

Approved For Release 2009/03/11 : CIA-RDP79-01083A000200070001-0

The semi-axes of the ellipsoid $\rho = \rho_0$ may now be expressed by the functions of Lamé according to the compiled table, thus:

$$A = R_1(\rho_0) = R_1^0;\; B = R_2(\rho_0) = R_2^0;\; C = R_3(\rho_0) = R_3^0.$$

The equation of Lamé is of the second order, so, besides the solution R, there must exist a second solution which we shall denote by S. Each function R is conjugate with another function, S, which is known as a function of Lamé of the second kind. These functions are determined as follows:

$$S = (2n + 1)\, R \int_{\rho}^{\infty} \frac{d}{R^2 W} . \qquad (W = R_1 R_2) \tag{13}$$

If n denotes the order of functions, the expansion of the function R into series starts from $\rho^n + \cdots$, and the expansion of the function S into series starts from $1/\rho^{n+1} + \cdots$. The function S is regular outside the surface of the ellipsoid ρ.

Therefore the potential function for the exterior region must be sought in the solution of the Stokes problem, which, in accordance with the above formulation, is in essence the exterior problem of Dirichlet, in the form

$$V = \sum_{k=0}^{\infty} A_k S_k M_k N.$$

The functions R and S are connected by the mutual relationship

$$S \frac{dP}{d\rho} - R \frac{dS}{d\rho} = \frac{2n+1}{W} . \tag{14}$$

The orthogonality of Lamé's function is expressed by the equality

$$\int_{E_0} l_0 MNM'N' \, d\omega = 0, \tag{15}$$

where the integration is the whole surface of the ellipsoid $E_0(\rho = \rho_0)$, l_0 is a magnitude introduced by Liouville for the ellipsoid ρ_0, and $d\omega$ is an element of the surface of the ellipsoid. It is to be noted that if the formula (15) we suppose $M'N' = M_0 N_0 = 1$, then the formula becomes

$$\int_{E_0} l_0 MN d\omega = 0. \qquad (16)$$

We must add also that

$$\int_{E_0} l_0 \, d\omega = 4\pi. \qquad (17)$$

We remark in conclusion that, in Lamé's equation, a transformation is possible to the elliptic variable

$$u = \int_{\rho}^{\infty} \frac{d\rho}{W} = S_0.$$

Lamé's equation then takes a simple form:

$$\frac{d^2R}{du^2} - \left[n(n+1)\wp(u) + h\right] R = 0,$$

where $\wp(u)$ is the function of Weierstrass.

3. THE POTENTIAL OF THE LEVEL TRIAXIAL ELLIPSOID; THE DETERMINATION OF THE COEFFICIENTS FROM THE EQUILIBRIUM CONDITIONS

Let us expand the potential function (potential) into Lamé's functions. As was noted in the preceding section, the expansion can be expressed by the equality

$$V = \sum_{0}^{\infty} A_k S_k M_k N_k \qquad (18)$$

On the surface of the ellipsoid this function must become

$$\text{const} - \frac{\omega^2}{2}(y^2 + z^2)$$

This expansion of the function V into products SMN must contain only those Lamé functions which become on the surface either constant or a polynomial in y^2, z^2 and also x^2, because the latter variable is a function of y^2 and z^2 on the surface where all the variable (x^2, y^2, z^2) are interrelated by the ellipsoid equation.

Approved For Release 2009/03/11 : CIA-RDP79-01083A000200070001-0

On the surface of the ellipsoid $\rho = \rho_0$ = const, the product SMN differs from the product RMN only by a constant multiplier because

$$SMN = \frac{S}{R} \ RMN,$$

and $\frac{S}{R}$ is constant if ρ is constant.

As for the products RMN, they can become, as is seen in the table of Lamé's functions (Table 2), either constant or polynomials in x^2, y^2, z^2 only in the case k = 0, 7, 8. The subsequent functions k = 9, 10, ... not included in the table do not have the required property.

All the coefficients in the expansion of (18) will be zero except A_0, A_7, A_8, and the expansion will have the form

$$V = A_0S_0M_0N_0 + A_7S_7M_7N_7 + A_8S_8M_8N_8. \qquad (19)$$

Besides, as was said earlier, the expansion of S starts with $\frac{1}{\rho^{n+1}}$, where n is the order of the function.

The expansion of S_0 starts with $\frac{1}{\rho}$ and S_7 and S_8 with $\frac{1}{\rho^3}$, and from the condition ($\lim_{z \to \infty} rV = M$) we obtain (taking into account that $M_0=N_0=1$):

$$A_0 = M$$

where M is the mass of the ellipsoid.

In this manner we obtain the final form of expansion of the potential of gravity in Lamé functions:

$$V = MS_0 + A_7S_7M_7N_7 + A_8S_8M_8N_8. \qquad (20)$$

The potential of the gravity U of the level triaxial ellipsoid rotating with constant angular velocity ω is

$$U = V + \frac{\omega^2}{2}(y^2 + z^2), \qquad (21)$$

Here the expression for V is as in (20).

It is necessary to expand $y^2 + z^2$ into the products MN in order to express the expansion of the gravity in Lamé functions.

Taking into account the example of Hamy, we shall use the formulae (5) that express the interrelationship among the rectangular and elliptic coordinates. Separating the terms depending on μ^2 and ν^2, the equations (5) can be written:

$$x^2 = \frac{\rho^2 - 1}{1 - \eta^2}\left[\mu^2\nu^2 - (\mu^2 + \nu^2) + 1\right],$$

$$y^2 = \frac{\rho^2 - \eta^2}{\eta^2(1 - \eta^2)}\left[- \mu^2\nu^2 + \eta^2(\mu^2 + \nu^2) - \eta^4\right], \qquad (22)$$

$$z^2 = \frac{\rho^2}{\eta^2}\mu^2\nu^2.$$

Let us now express $\mu^2\nu^2$ and $\mu^2 + \nu^2$ in terms of the products M_7N_7 and M_8N_8.

For this purpose we shall write

$$M_7N_7 = (\mu^2 - \alpha_1)(\nu^2 - \alpha_1) = \mu^2\nu^2 - \alpha_1(\mu^2 + \nu^2) + \alpha_1^2,$$
$$\qquad (23)$$
$$M_8N_8 = (\mu^2 - \alpha_2)(\nu^2 - \alpha_2) = \mu^2\nu^2 - \alpha_2(\mu^2 + \nu^2) + \alpha_2^2.$$

We obtain of these equalities,

$$\mu^2\nu^2 = \frac{\alpha_1M_8N_8 - \alpha_2M_7N_7}{\alpha_1 - \alpha_2} + \alpha_1\alpha_2,$$

$$\mu^2 + \nu^2 = \frac{M_8N_8 - M_7N_7}{\alpha_1 - \alpha_2} + (\alpha_1 + \alpha_2).$$

Setting the values of $\alpha_1\alpha_2$ and $\alpha_1 + \alpha_2$ from equation (10), we obtain

$$\mu^2\nu^2 = \frac{\alpha_1M8N8 - \alpha_2M_7N_7}{\alpha_1 - \alpha_2} + \frac{\eta^2}{3},$$

$$\mu^2 + \nu^2 = \frac{M_8N_8 - M_7N_7}{\alpha_1 - \alpha_2} + \frac{2}{3}(1 + \eta^2).$$

Setting these expressions into (22), we obtain, after some modifications,

$$x^2 = \frac{\rho^2 - 1}{1 - \eta^2}\left[\frac{1 - \eta^2}{3} + \frac{1 - \alpha_2}{\alpha_1 - \alpha^2} M_7N_7 + \frac{1 - \alpha_1}{\alpha_2 - \alpha_1} M_8N_8\right],$$

$$y^2 = \frac{\rho^2 - \eta^2}{\eta^2(1 - \eta^2)}\left[\frac{\eta^2(1 - \eta^2)}{3} + \frac{\eta^2 - \alpha_2}{\alpha_2 - \alpha_1} M_7N_7 + \frac{\eta^2 - \alpha_1}{\alpha_1 - \alpha_2} M_8N_8\right],$$

$$z^2 = \frac{\rho^2}{\eta^2}\left[\frac{\eta^2}{3} + \frac{\alpha_2}{\alpha_2 - \alpha_1} M_7N_7 + \frac{\alpha_1}{\alpha_1 - \alpha_2} M_8N_8\right].$$

It is easy to get the following identities taking into account the interrolationship of the coefficients and roots of the quadratic equation (10):

$$\begin{aligned} \eta^2 &= 3\alpha_1\alpha_2, \\ (1 - \eta^2) &= 3(1 - \alpha_1)(1 - \alpha_2), \\ \eta^2(1 - \eta^2) &= 3(\alpha_1 - \eta^2)(\eta^2 - \alpha_2). \end{aligned} \tag{24}$$

By aid of these identities the following expressions for x^2, y^2, and z^2 can be formulated:

$$\begin{aligned} x^2 &= (\rho^2 - 1)\left[\frac{1}{3} + \frac{1}{3(\alpha_1 - 1)}\frac{M_7N_7}{\alpha_2 - \alpha_1} + \frac{1}{3(\alpha_2 - 1)}\frac{M_8N_8}{\alpha_1 - \alpha_2}\right], \\ y^2 &= (\rho^2 - \eta^2)\left[\frac{1}{3} + \frac{1}{3(\alpha_1 - \eta^2)}\frac{M_7N_7}{\alpha_2 - \alpha_1} + \frac{1}{3(\alpha_2 - \eta^2)}\frac{M_8N_8}{\alpha_1 - \alpha_2}\right], \\ z^2 &= \rho^2\left[\frac{1}{3} + \frac{1}{3\alpha_1}\frac{M_7N_7}{\alpha_2 - \alpha_1} + \frac{1}{3\alpha_2}\frac{M_8N_8}{\alpha_1 - \alpha_2}\right], \end{aligned} \tag{25}$$

and

$$y^2 + z^2 = \frac{2\rho^2 - \eta^2}{3} + \frac{M_7N_7}{3(\alpha_2 - \alpha_1)}\left[\frac{\rho^2 - \eta^2}{\alpha_1 - \eta^2} \quad \frac{\rho^2}{\alpha_1}\right] + \frac{M_8N_8}{3(\alpha_1 - \alpha_2)}\left[\frac{\rho^2 - \eta^2}{\alpha_2 - \eta^2} + \frac{\rho^2}{\alpha_2}\right].$$

Let us modify the expressions in the brackets:

$$\frac{\rho^2 - \eta^2}{\alpha_1 - \eta^2} + \frac{\rho^2}{\alpha_1} = \frac{\rho^2(2\alpha_1 - \eta^2) - \alpha_1\eta^2}{\alpha_1(\alpha_1 - \eta^2)} .$$

Let us add and subtract, in the numerator, the expression $\alpha_1(2\alpha_1 - \eta^2)$. Then we can write

$$\frac{(2\alpha_1 - \eta^2)(\rho^2 - \alpha_1) + 2\alpha_1^2 - 2\alpha_1\eta^2}{\alpha_1(\alpha_1 - \eta^2)} = \frac{(2\alpha_1 - \eta^2)(\rho^2 - \alpha_1)}{\alpha_1(\alpha_1 - \eta^2)} + 2.$$

Let us eliminate α_2 from the second identity of (24) by aid of the first identity, then

$$\alpha_1(\alpha_1 - \eta^2) = (1 - \alpha_1)(2\alpha_1 - \eta^2),$$

and finally

$$\frac{\rho^2 - \eta^2}{\alpha_1 - \eta^2} + \frac{\rho^2}{\alpha_1} = \frac{\rho^2 - \alpha_1}{1 - \alpha_1} + 2.$$

By analogous operations the second bracket becomes

$$\frac{\rho^2 - \eta^2}{\alpha_2 - \eta^2} + \frac{\rho^2}{\alpha_2} = \frac{\rho^2 - \alpha_2}{1 - \alpha_2} + 2.$$

By aid of the expressions given in Table 2 for the Lamé functions, these two equations can be written in the form:

$$\frac{R_2^2}{\alpha_1 - \eta^2} + \frac{R_3^2}{\alpha_1} = \frac{R_7}{1 - \alpha_1} + 2 \tag{26}$$

$$\frac{R_2^2}{\alpha_2 - \eta^2} + \frac{R_3^2}{\alpha_2} = \frac{R_8}{1 - \alpha_2} + 2.$$

Finally, we get for $y^2 + z^2$ the expression

$$y^2 + z^2 = \frac{2\rho^2 - \eta^2}{3} + \frac{M_7N_7}{3(\alpha_2 - \alpha_1)}\left[\frac{R_7}{1 - \alpha_1} + 2\right] + \frac{M_8N_8}{3(\alpha_1 - \alpha_2)}\left[\frac{R_8}{1 - \alpha_2} + 2\right]. \qquad (26.1)$$

Setting this formula into the formula (21), we have

$$U = MS_0 + A_7S_7M_7N_7 + A_8S_8M_8N +$$

$$+ \frac{\omega^2}{2}\left\{\frac{2\rho^2 - \eta^2}{3} + \frac{M_7N_7}{3(\alpha_2 - \alpha_1)}\left[\frac{R_7}{1 - \alpha_1} + 2\right] + \frac{M_8N_8}{3(\alpha_1 - \alpha_2)}\left[\frac{R_8}{1 - \alpha_2} + 2\right]\right\}.$$

Finally, from this we obtain the following transformed form of the gravitational potential:

$$U = MS_0 + \frac{\omega^2}{6}(2\rho^2 - \eta^2) + \left\{A_7S_7 + \frac{\omega^2}{6(\alpha_2 - \alpha_1)}\left[\frac{R_7}{1 - \alpha_1} + 2\right]\right\} M_7N_7 +$$

$$+ \left\{A_8S_8 + \frac{\omega^2}{6(\alpha_1 - \alpha_2)}\left[\frac{R_8}{1 - \alpha_2} + 2\right]\right\} M_8N_8. \qquad (27)$$

On the surface of the ellipsoid, so that $\rho = \rho_0$

$$\left\{MS_0^0 + \frac{\omega^2}{6}(2\rho_0^2 - \eta^2)\right\} M_0N_0 + \left\{A_7S_7^0 + \frac{\omega^2}{6(\alpha_2 - \alpha_1)}\left[\frac{R_7^0}{1 - \alpha_1} + 2\right]\right\} M_7N_7 +$$

$$+ \left\{A_8S_8^0 + \frac{\omega^2}{6(\alpha_1 - \alpha_2)}\left[\frac{R_8^0}{1 - \alpha_2} + 2\right]\right\} M_8N_8 = \text{const.} \qquad (27.1)$$

The last expression becomes a constant on the surface of an ellipsoid $\rho = \rho_0$ if the coefficients of M_7N_7 and M_8N_8 are zero. Actually, we have an expansion of a constant in Lamé functions in the form

$$C = a_0L_0 + a_7L_7 + a_8L_8, \qquad (28)$$

where for an abbreviation the product MN is denoted by L, and the coefficients a_0, a_7, a_8 denote the expressions in braces, which are constant on the surface of the ellipsoid.

If the equation be multiplied by $l_0 d\omega$ and integrated over the surface of the ellipsoid $E_0(\rho = \rho_0)$, we have

$$C \int_{E_0} l_0 d\omega = a_0 \int_{E_0} l_0 L_0 d\omega + a_7 \int_{E_0} l_0 L_7 d\omega + a_8 \int_{E_0} l_0 L_8 d\omega.$$

The second and the third terms on the right side are zero on the basis of the orthogonality of the Lamé functions (16). Taking into account the equation (17) the integral formula becomes

$$C\, 4\pi = a_0\, 4\pi,$$

or

$$C = a_0.$$

We draw the conclusion that the constant which expresses the potential of the gravity on the surface of the ellipsoid has the form

$$C = MS_0^{\ 0} + \frac{\omega^2}{6}\left(2\rho_0^{\ 2} - \eta^2\right). \tag{29}$$

In order to prove that the coefficients a_7 and a_8 are equal to zero, it is sufficient to multiply (28) in turn by $\ell_0 L_7 d\omega$ and $\ell_0 L_8 d\omega$ and to integrate the results over the surface of the ellipsoid. Then, from equations (15) and (16), the equality (28) gives

$$a_7 \int_{E_0} \ell_0 L_7^{\ 2} d\omega = 0,$$

$$a_8 \int_{E_0} \ell_0 L_8^{\ 2} d\omega = 0.$$

The integrals are not equal to zero; consequently

$$a_7 = 0,$$
$$a_8 = 0,$$

or

$$A_7 = \frac{\omega^2}{6(\alpha_1 - \alpha_2)S_7^{\ 0}}\left[\frac{R_7^{\ 0}}{1 - \alpha_1} + 2\right],$$

$$A_8 = \frac{\omega^2}{6(\alpha_2 - \alpha_1)S_8^{\ 0}}\left[\frac{R_8^{\ 0}}{1 - \alpha_2} + 2\right]. \tag{30}$$

These equalities determine the conditions of equilibrium.

Setting these expressions for the coefficients into the formula (20), we get the final expression of the gravitational potential on the level triaxial ellipsoid:*

$$V = MS_0 + \frac{\omega^2}{6(\alpha_1 - \alpha_2)S_7^0}\left[\frac{R_7^0}{1 - \alpha_1} + 2\right] S_7M_7N_7 + \\ + \frac{\omega^2}{6(\alpha_2 - \alpha_1)S_8^0}\left[\frac{R_8^0}{1 - \alpha_2} + 2\right] S_8M_8N_8. \qquad (31)$$

4. THE CASE OF A HOMOGENEOUS LEVEL ELLIPSOID

We need in the further investigations some identities by whose aid the functions S_7 and S_8 could be expressed in terms of the functions S_1, S_2, S_3. Let us furnish a proof of the existence of such an identity for the function S_7. A like proof exists for the function S_8.**

We transform to a new variable $\rho^2 = r$. Then $\rho d\rho = \frac{1}{2}dr$, and the function S_7, as a function of the variable r, will have the form

$$S_7(r) = \frac{5}{2} R_7(r) \int_r^\infty \frac{dr}{R_7^2(r)F(r)} , \qquad (32)$$

and

$$F(r) = r(r - \gamma^2)(r - 1).$$

Let us integrate (32) by parts.

$$\frac{S_7(r)}{5R_7(r)} = \frac{1}{2}\left[- \frac{1}{F(r)R_7(r)}\right]_r^\infty - \frac{1}{2}\int_r^\infty \frac{F'(r)}{R_7(r)\ F(r)} \frac{dr}{F(r)} . \qquad (33)$$

$F'(r)$ denotes the result of the differentiation with regard to r, and $R_7'(r) = 1$. Let us expand the function $\frac{F'(r)}{R_7(r)F(r)}$ into simple fractions.

* An analogous expression can be found at Mineo, see (10).
** The proofs of the identity for a general case (for any n) we find in the work of Humbert (19).

We have

$$\frac{F'(r)}{R_7(r)F(r)} = \frac{F'(r)F(r)}{R_7(r)F^2(r)} = \frac{1}{2}\left[\frac{1}{R_7(0)}\frac{1}{r} + \frac{1}{R_7(\eta^2)}\frac{1}{r-\eta^2} + \frac{1}{R_7(1)}\frac{1}{r-1}\right]. \qquad (34)$$

Indeed, we have the expansion of a function of the form $\frac{\phi(r)}{\psi(r)}$, where $\phi(r) = F'(r)F(r)$, and $\psi(r) = R_7(r)F^2(r) = (r-\alpha_1)(r-1)(r-\eta^2)r$. This expansion can be written

$$\frac{\phi(r)}{\psi(r)} = \frac{\phi(0)}{\psi'(0)}\frac{1}{r} + \frac{\phi(1)}{\psi'(1)}\frac{1}{r-1} + \frac{\phi(\eta^2)}{\psi'(\eta^2)}\frac{1}{r-\eta^2} + \frac{\phi(\alpha_1)}{\psi'(\alpha_1)}\frac{1}{r-\alpha_1}.$$

But $\psi'(r) = 2R_7(r)F'(r)F(r) + F^2(r)$, where $F^2(r) = \left[F(r)\right]^2$.

The coefficients of the expansion have the form

$$\frac{F'(r)}{2\,R_7(r)F'(r) + F(r)}.$$

If $r = 0, \eta, 1$, the second term is equal to zero, and we have for the first, second, and third coefficients the corresponding values,

$$\frac{1}{2R_7(0)},\ \frac{1}{2R_7(\eta^2)},\ \text{and}\ \frac{1}{2R_7(1)}.$$

If $r = \alpha_1$, the first term in the denominator is equal to zero, and we have for the calculation of the last coefficient the formula

$$\frac{F'(r)}{F(r)} = \frac{1}{2}\left[\frac{1}{r} + \frac{1}{r-\eta^2} + \frac{1}{r-1}\right], \qquad (34.1)$$

which is equal to zero if $r = \alpha_1$, according to (9).

Setting (34) into (33), we have*

* This is a partial case of a common formula obtained by Humbert (see (19) page 36) in which it is necessary to correct a mistake in the coefficient before the integral. There it is given as 1/2, but it must be 1/4.

$$\frac{S_7(r)}{5R_7(r)} = \frac{1}{2R_7(r)F(r)} - \frac{1}{4}\int_r^\infty \left[\frac{1}{R_7(0)}\frac{1}{r} + \frac{1}{R_7(\eta^2)}\frac{1}{r-\eta^2} + \frac{1}{R_7(1)}\frac{1}{r-1}\right]\frac{dr}{F(r)}, \tag{35}$$

or

$$\frac{S_7(r)}{5R_7(r)} = \frac{1}{2R_7(r)F(r)} + \frac{1}{4}\int_r^\infty \left[\frac{1}{\alpha_1}\frac{1}{r} + \frac{1}{\alpha_1-\eta^2}\frac{1}{r-\eta^2} + \frac{1}{\alpha_1-1}\frac{1}{r-1}\right]\frac{dr}{F(r)}. \tag{36}$$

Let us write the expressions for S_1, S_2, S_3:

$$S_1 = \frac{3}{2}R_1(r)\int_r^\infty \frac{dr}{R_1^2(r)F(r)},$$

$$S_2 = \frac{3}{2}R_2(r)\int_r^\infty \frac{dr}{R_2^2(r)F(r)},$$

$$S_3 = \frac{3}{2}R_3(r)\int_r^\infty \frac{dr}{R_3^2(r)F(r)}.$$

Hence

$$\int_r^\infty \frac{dr}{(r-1)F(r)} = \frac{2}{3}\frac{S_1(r)}{R_1(r)},$$

$$\int_r^\infty \frac{dr}{(r-\eta^2)F(r)} = \frac{2}{3}\frac{S_2(r)}{R_2(r)},$$

$$\int_r^\infty \frac{dr}{rF(r)} = \frac{2}{3}\frac{S_3(r)}{R_3(r)}.$$

From the equality (36) we obtain the form

$$\frac{S_7(r)}{5R_7(r)} = \frac{1}{2R_7(r)F(r)} + \frac{1}{6}\left[\frac{1}{\alpha_1}\frac{S_3(r)}{R_3(r)} + \frac{1}{\alpha_1 - \eta^2}\frac{S_2(r)}{R_2(r)} + \frac{1}{\alpha_1 - 1}\frac{S_1(r)}{R_1(r)}\right].$$

So we obtain the identity

$$2\frac{S_7}{R_7} - \frac{5}{\Delta R_7} = \frac{5}{3}\left[\frac{1}{\alpha_1}\frac{S_3}{R_3} + \frac{1}{\alpha_1 - \eta^2}\frac{S_2}{R_2} + \frac{1}{\alpha_1 - 1}\frac{S_1}{R_1}\right], \tag{36.1}$$

Here it makes no difference which variable is taken, r or ρ, and Δ is here the product of $R_1R_2R_3$.

We have an analogous expression for S_8:

$$2\frac{S_8}{R_8} - \frac{5}{\Delta R_8} = \frac{5}{3}\left[\frac{1}{\alpha_2}\frac{S_3}{R_3} + \frac{1}{\alpha_2 - \eta^2}\frac{S_2}{R_2} + \frac{1}{\alpha^2 - 1}\frac{S_1}{R_1}\right]. \tag{36.2}$$

Let us obtain one more identity by writing the identity (36.1) in another form. Taking into account that $\rho = R_3$, and that $W = R_1R_2$, we have after multiplying the equality (36.1) by $R_3 = \rho$:

$$2\rho\frac{S_7}{R_7} - \frac{5}{WR_7} = \frac{5}{6}2\rho\left[\frac{1}{\alpha_1}\frac{S_3}{R_3} + \frac{1}{\alpha_1 - \eta^2}\frac{S_2}{R_2} + \frac{1}{\alpha_1 - 1}\frac{S_1}{R_1}\right].$$

Now

$$\frac{d}{d\rho}S_7 = \frac{5}{6}\frac{d}{d\rho}\left[\frac{R_3S_3}{\alpha_1} + \frac{R_2S_2}{\alpha_1 - \eta^2} + \frac{R_1S_1}{\alpha_1 - 1}\right] \tag{37}$$

Indeed

$$\frac{d}{d\rho} S_3R_3 = 2\rho \int_0^\infty \frac{3d\rho}{\rho^2 W} - \frac{3}{W} = 2\rho \frac{S_3}{R_3} - \frac{3}{W},$$

$$\frac{d}{d\rho} S_2R_2 = 2\rho \int_0^\infty \frac{3d\rho}{(\rho^2 - \eta^2)W} - \frac{3}{W} = 2\rho \frac{S_2}{R_2} - \frac{3}{W}, \qquad (37.1)$$

$$\frac{d}{d\rho} S_1R_1 = 2\rho \int_0^\infty \frac{3d}{(\rho^2 - 1)\,W} - \frac{3}{W} = 2\rho \frac{S_1}{R_1} - \frac{3}{W}.$$

Multiplying these equalities in turn by $\frac{1}{\alpha_1}$, $\frac{1}{\alpha_1 - \eta^2}$, $\frac{1}{\alpha_1 - 1}$, adding the results, and taking into account that according to (9)

$\frac{1}{\alpha_1} + \frac{1}{\alpha_1 - \eta^2} + \frac{1}{\alpha_1 - 1} = 0$, we obtain

$$\frac{d}{d\rho}\left[\frac{R_3S_3}{\alpha_1} + \frac{R_2S_2}{\alpha_1 - \eta^2} + \frac{R_1S_1}{\alpha_1 - 1}\right] = 2\rho\left[\frac{1}{\alpha_1}\frac{S_3}{R_3} + \frac{1}{\alpha_1 - \eta^2}\frac{S_2}{R_2} + \frac{1}{\alpha_1 - 1}\frac{S_1}{R_1}\right]. \qquad (37.2)$$

In accordance with (14):

$$\frac{dS_7}{d} = 2\rho \frac{S_7}{R_7} - \frac{5}{WR_7}. \qquad (37.3)$$

Taking into account the equations (37.2) and (37.3), we see that the equation (37) is correct. The integration of (37) between the limits from ρ to ∞ gives*

$$\begin{aligned} S_7 &= \frac{5}{6}\left[\frac{R_3S_3}{\alpha_1} + \frac{R_2S_2}{\alpha_1 - \eta^2} + \frac{R_1S_1}{\alpha_1 - 1}\right] \\ S_8 &= \frac{5}{6}\left[\frac{R_3S_3}{\alpha_2} + \frac{R_2S_2}{\alpha_2 - \eta^2} + \frac{R_1S_1}{\alpha_2 - 1}\right]. \end{aligned} \qquad (38)$$

* Identity (38) may be found in the cited memoire of Hamy(9), but he arrives at it in an entirely different manner.

Now we shall prove that the conditions of equilibrium obtained (30) in the proceding section can be reduced to the equilibrium conditions on homogenous level triaxial ellipsoids of Jacobi if the coefficients A_7 and A_8 are equal to

$$A_7 = \frac{T}{5(\alpha_1 - \alpha_2)}, \qquad (39)$$

$$A_8 = \frac{T}{5(\alpha_2 - \alpha_1)},$$

where T is the volume of the ellipsoid.

We shall prove in the end of this section that the expression for the attraction potential (20) can be reduced to the expression of the potential of the homogeneous ellipsoid of Dirichlet obtained in the year 1846. (21)

The conditions (30) can be written with the values of the coefficients (39) in the form:

$$\frac{T}{5(\alpha_1 - \alpha_2)} = \frac{\omega^2}{6S_7(\alpha_1 - \alpha_2)}\left[\frac{R_8}{1 - \alpha_1} + 2\right],$$

$$\frac{T}{5(\alpha_2 - \alpha_1)} = \frac{\omega^2}{6S_8(\alpha_2 - \alpha_1)}\left[\frac{R_8}{1 - \alpha_2} + 2\right],$$

Hence

$$TS_7 = \frac{5}{6}\omega^2\left[\frac{R_7}{1 - \alpha_1} + 2\right],$$

$$TS_8 = \frac{5}{6}\omega^2\left[\frac{8}{1 - \alpha_2} + 2\right].$$

Taking the sum and difference of the preceding expressions we obtain

$$T(S_7 + S_8) = \frac{5}{6}\omega^2\left[\frac{R_7}{1 - \alpha_1} + \frac{R_8}{1 - \alpha_2} + 4\right],$$

$$T(S_7 - S_8) = \frac{5}{6}\omega^2\left[\frac{R_7}{1 - \alpha_1} - \frac{R_8}{1 - \alpha_2}\right].$$

Consequently

$$\frac{5}{6}\frac{\omega^2}{T} = \frac{S_7 + S_8}{\dfrac{R_7}{1 - \alpha_1} + \dfrac{R_8}{1 - \alpha_2} + 4}, \tag{40.1}$$

$$\frac{5}{6}\frac{\omega^2}{T} = \frac{S_7 + S_8}{\dfrac{R_7}{1 - \alpha_1} - \dfrac{R_8}{1 - \alpha_2}}. \tag{40.2}$$

Let us transform the right side of the equations (40) using the identities (38) and (26).

According to the identity (38):

$$S_7 + S_8 = \frac{5}{6}\left\{R_3S_3\left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right] + R_2S_2\left[\frac{1}{\alpha_1 - \eta^2} + \frac{1}{\alpha_2 - \eta^2}\right] + R_1S_1\left[\frac{1}{\alpha_1 - 1} + \frac{1}{\alpha_2 - 1}\right]\right\},$$

Taking into account the quadratic equation (10), whose roots $\alpha_1 + \alpha_2 = \frac{2}{3}(1 + \eta^2)$, and the identity (24), we obtain

$$S_7 + S_8 = \frac{5}{6}\left\{R_3S_3\left[\frac{2(1 + \eta^2)}{\eta^2}\right] + R_2S_2\left[\frac{4\eta^2 - 2}{\eta^2(1 - \eta^2)}\right] + R_1S_1\left[\frac{2\eta^2 - 4}{1 - \eta^2}\right]\right\}.$$

Making some transformations, we get

$$S_7 + S_8 = \frac{5}{3\eta^2(1 - \eta^2)}\left\{R_3S_3(1 - \eta^2) + R_2S_2(2\eta^2 - 1) + R_1S_1\eta^2(\eta^2 - 2)\right\}$$

and

$$\frac{R_7}{1 - \alpha_1} + \frac{R_8}{1 - \alpha_2} + 4 = R_2^2\left[\frac{1}{\alpha_1 - \eta^2} \quad \frac{1}{\alpha_2 - \eta^2}\right] + R_3^2\left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right].$$

Using the above-mentioned identities, we can reduce the latter equation to the form

$$\frac{R_7}{1 - \alpha_1} + \frac{R_8}{1 - \alpha_2} + 4 = \frac{2}{\eta^2(1 - \eta^2)}\left[R_2^2(2\eta^2 - 1) + R_3^2(1 - \eta^4)\right].$$

Analogous computations of the numerator and denominator of (40.2) give

$$S_7 - S_8 = \frac{5(\alpha_2 - \alpha_1)}{2\eta^2(1-\eta^2)}\left[R_3S_3(1-\eta^2) - R_2S_2 - R_1S_1\eta^2\right],$$

and

$$\frac{R_7}{1-\alpha_1} - \frac{R_8}{1-\alpha_2} = \frac{3(\alpha_2 - \alpha_1)}{\eta^2(1-\eta^2)}\left[R_3{}^2(1-\eta^2) - R_2{}^2\right].$$

The right sides of the equation (40) get the form

$$\frac{5}{6}\,\frac{R_3S_3(1-\eta^2) - R_2S_2 + R_1S_1\eta^2}{R_3{}^2(1-\eta^2) - R_2{}^2},$$

$$\frac{5}{6}\,\frac{R_3S_3(1-\eta^4) - R_2S_2(1-2\eta^2) + R_1S_1\eta^2(\eta^2-2)}{R_3{}^2(1-\eta^4) - R_2{}^2(1-2\eta^2)}.$$

Setting these expressions into (40), we obtain

$$\frac{\omega^2}{T} = \frac{R_3S_3(1-\eta^2) - R_2S_2 - R_1S_1\eta^2}{R_3{}^2(1-\eta^2) - R_2{}^2},$$

$$\frac{\omega^2}{T} = \frac{R_3S_3(1-\eta^4) - R_2S_2(1-2\eta^2) + R_1S_1\eta^2(\eta^2-2)}{R_3{}^2(1-\eta^4) + R_2{}^2(1-2\eta^2)} \tag{41}$$

Both the conditions can be reduced to the conditions of equilibrium of Maclaurin ellipsoids in the case of rotation ellipsoids.

Indeed, let us suppose in (41) $R_3 = R_2$; then we have:

$$\frac{\omega^2}{T} = \frac{-\eta^2R_2S_2 + \eta^2R_1S_1}{-\eta^2R_2{}^2} = \frac{R_2S_2 - R_1S_1}{R_2{}^2},$$

$$\frac{\omega^2}{T} = \frac{R_2S_2(2\eta^2 - \eta^4) + R_1S_1\eta^2(\eta^2-2)}{R_2{}^2(-\eta^4 + 2\eta^2)} = \frac{R_2S_2 - R_1S_1}{R_2{}^2}.$$

The comparison of the parts of the equality (41) on the right side gives, after the deduction of similar terms:

$$R_3S_3R_2^{\,2}\left[(1-\eta^4)-(1-\eta^2)(1-2\eta^2)\right]-R_1S_1R_2^{\,2}\left[\eta^2(1-2\eta^2)-\eta^2(\eta^2-2)\right]=$$
$$=R_2S_2R_3^{\,2}\left[(1-\eta^4)-(1-2\eta^2)(1-\eta^2)\right]-$$
$$-R_1S_1R_3^{\,2}\left[\eta^2(1-\eta^4)-\eta^2(\eta^2-2)(1-\eta^2)\right].$$

All the expressions in the brackets are equal to $3\eta^2 - 3\eta^4$.

Dividing through by this nonzero expression we obtain:

$$R_3S_3R_2^{\,2}-R_1S_1R_2^{\,2}=R_2S_2R_3^{\,2}-R_1S_1R_3^{\,2},$$
$$\frac{R_3S_3-R_1S_1}{R_3^{\,2}}=\frac{R_2S_2-R_1S_1}{R_2^{\,2}}. \tag{42}$$

This expression is the above-mentioned supplementary condition of the equilibrium of Jacobi ellipsoids which was given by Poincaré (20) in the form:

$$\frac{R_1S_1}{3}=\frac{R_4S_4}{5}. \tag{43}$$

The equilibrium conditions (30) can be reduced to the equilibrium conditions of Jacobi ellipsoids if the coefficients are equal to the expressions (39).

Let us prove now that the substitution into the formula (20) of the values of the coefficients expressed in (39) makes the formula (20) equivalent to the formula of Dirichlet.

The formula (20) after this operation becomes[1]

$$V=\left(T\,S_0+\frac{1}{5(\alpha_1-\alpha_2)}\,S_7M_7N_7+\frac{1}{5(\alpha_2-\alpha_1)}\,S_8M_8N_8\right). \tag{44}$$

Let us introduce into the last formula rectangular coordinates of an exterior point x, y, z.

1. The density becomes equal to unity, i.e., $M = T_0$.

We had the expressions for x^2, y^2, z^2 in terms of the product, MN, namely the formulas (25). Let us write the formulae using the expressions of Lamé from Table 2:

$$x^2 = \frac{R_1^{\,2}}{3} + \frac{R_1^{\,2}}{3(\alpha_1 - 1)} \frac{M_7N_7}{\alpha_2 - \alpha_1} + \frac{R_1^{\,2}}{3(\alpha_2 - 1)} \frac{M_8\, N_8}{\alpha_1 - \alpha_2} ,$$

$$y^2 = \frac{R_2^{\,2}}{3} + \frac{R_2^{\,2}}{3(\alpha_1 - \eta^2)} \frac{M_7N_7}{\alpha_2 - \alpha_1} + \frac{R_2^{\,2}}{3(\alpha_2 - \eta^2)} \frac{M_8N_8}{\alpha_1 - \alpha_2} ,$$

$$z^2 = \frac{R_3^{\,2}}{3} + \frac{R_3^{\,2}}{3\alpha_1} \frac{M_7N_7}{\alpha_2 - \alpha_1} + \frac{R_3^{\,2}}{3\alpha_2} \frac{M_8N_8}{\alpha_1 - \alpha_2} .$$

Hence

$$1 - \frac{3x^2}{R_1^{\,2}} = \frac{1}{\alpha_1 - \alpha_2}\left(\frac{M_7N_7}{\alpha_1 - 1} - \frac{M_8N_8}{\alpha_2 - 1}\right),$$

$$1 - \frac{3y^2}{R_2^{\,2}} = \frac{1}{\alpha_1 - \alpha_2}\left(\frac{M_7N_7}{\alpha - \eta^2} - \frac{M_8N_8}{\alpha_2 - \eta^2}\right) , \tag{45}$$

$$1 - \frac{3z^2}{R_3^{\,2}} = \frac{1}{\alpha_1 - \alpha_2}\left(\frac{M_7N_7}{\alpha_1} - \frac{M_8N_8}{\alpha_2}\right).$$

Let us put into formula (44) instead of S_7 and S_8 their expressions by (38):

$$V = T\left[S_0 + \frac{R_1S_1}{6(\alpha_1 - \alpha_2)}\left(\frac{M_7N_7}{\alpha_1 - 1} - \frac{M_8N_8}{\alpha_2 - 1}\right) + \frac{R_2S_2}{6(\alpha_1 - \alpha_2)}\left(\frac{M_7N_7}{\alpha_1 - \eta^2} - \frac{M_8N_8}{\alpha_2 - \eta^2}\right) + \frac{R_3S_3}{6(\alpha_1 - \alpha_2)}\left(\frac{M_7N_7}{\alpha_1} - \frac{M_8N_8}{\alpha_2}\right)\right].$$

Utilizing the equations (45) we obtain,

$$V = T\left[S_0 + \frac{R_1S_1}{6}\left(1 - \frac{3x^2}{R_1^{\,2}}\right) + \frac{R_2S_2}{6}\left(1 - \frac{3y^2}{R_2^{\,2}}\right) + \frac{R_3R_3}{6}\left(1 - \frac{3z^2}{R_3^{\,2}}\right)\right] ,$$

or

$$V = T\left[S_0 + \frac{1}{6}\,(R_1S_1 + R_2S_2 + R_3S_3) - \frac{S_1}{2R_1}\,x^2 - \frac{S_2}{2R_2}\,y^2 - \frac{S_3}{2R_3}\,z^2\right]. \tag{46}$$

Let us now prove the identity

$$\frac{1}{3}\left(R_1S_1 + R_2S_2 + R_3S_3\right) = S_0.$$

We shall study for this purpose the function

$$S_3 = 3R_3 \int_\rho^\infty \frac{d\rho}{R_3{}^2 W}$$

or

$$\frac{S_3}{3R_3} = \int_\rho^\infty \frac{d\rho}{\rho^2 W} = \int_\rho^\infty \frac{1}{W}\, d\left(-\frac{1}{\rho}\right).$$

Integrating this formula by parts, we get

$$\frac{S_3}{3R_3} = \left[-\frac{1}{\rho W}\right]_\rho^\infty - \int_\rho^\infty \frac{W' d\rho}{\rho W^2} = \frac{1}{\rho W} - \int_\rho^\infty \frac{W'}{\rho W}\,\frac{d\rho}{W} \tag{47}$$

Let us expand the function $\frac{W'}{\rho W}$ into simple fractions:

$$\frac{W'}{\rho W} = \frac{W'W}{\rho W^2} = \frac{2\rho^2 - 1 - \eta^2}{(\rho^2 - 1)(\rho^2 - \eta^2)} = \frac{1}{\rho^2 - 1} + \frac{1}{\rho^2 - \eta^2},$$

or

$$\frac{W'}{\rho W} = \frac{1}{R_1{}^2} + \frac{1}{R_2{}^2} \quad .$$

Setting this expression into (47) under the integral sign we obtain

$$\frac{S_3}{3R_3} = \frac{1}{\rho W} - \int_\rho^\infty \left[\frac{1}{R_1{}^2} + \frac{1}{R_2{}^2}\right]\frac{d\rho}{W}\ .$$

But S_1 and S_2 are :

$$S_1 = 3R_1 \int_\rho^\infty \frac{d\rho}{R_1{}^2 W},$$

$$S_2 = 3R_2 \int_\rho^\infty \frac{d\rho}{R_2{}^2 W}\ .$$

Taking this into account, we have

$$\frac{S_3}{3R_3} = \frac{1}{\rho W} - \frac{S_2}{3R_2} - \frac{S_1}{3R_1},$$

or (48)

$$\rho\left[\frac{S_3}{R_3} + \frac{S_2}{R_2} + \frac{S_1}{R_1}\right] = \frac{3}{W}.$$

It is easy to see that the addition of the expressions (37.1) gives

$$\frac{d}{d\rho}\left[R_3S_3 + R_2S_2 + R_1S_1\right] = 2\rho\left[\frac{S_3}{R_3} + \frac{S_2}{R_2} + \frac{S_1}{R_1}\right] - \frac{9}{W}, \tag{49}$$

besides

$$\frac{d}{d\rho} S_0 = -\frac{1}{W}. \tag{50}$$

Consequently, on the basis of (49) and (50) we can write

$$\frac{1}{2}\frac{d}{d\rho}\left[R_3S_3 + R_2S_2 + R_1S_1\right] = \frac{3}{2}\frac{d}{d\rho} S_0 .$$

Integrating this expression between the limits ρ to ∞, we obtain the sought identity:

$$\frac{1}{3}\left(R_3S_3 + R_2S_2 + R_1S_1\right) = S_0. \tag{51}$$

Using the identity (51), we can write (47) in the form

$$V = \frac{3}{2} T\left(S_0 - \frac{S_1}{3R_1} x^2 - \frac{S_2}{3R_2} y^2 - \frac{S_3}{3R_3} z^2\right), \tag{52}$$

or, substituting for S_1, S_2, S_3 their integral expressions,

$$V = \frac{3}{2} T \int_t^{\infty} \left(1 - \frac{x^2}{t^2 - 1} - \frac{y^2}{t^2 - \eta^2} - \frac{z^2}{t^2}\right) \frac{dt}{\sqrt{(t^2 - \eta^2)(t^2 - 1)}} .$$

Let us introduce a new variable

$$u = t^2 - \rho_0^2 \quad (\rho_0 \text{ is a constant}),$$

then

$$du = 2t^2 dt; \quad t = \sqrt{\rho_0^2 + u},$$

$$dt = \frac{du}{2\sqrt{\rho_0^2 + u}}.$$

Set $\sqrt{\rho_0^2 + u} = \lambda$.

The transformation to the new variable u gives

$$V = \frac{3}{4} T \int_{\lambda}^{\infty} \left(1 - \frac{x^2}{\rho_0^2 - 1 + u} - \frac{y^2}{\rho_0^2 - \eta^2 + u} - \frac{z^2}{\rho_0^2 + u} \right) \times$$

$$\times \frac{du}{\sqrt{(\rho_0^2 + u)(\rho_0^2 - \eta^2 + u)(\rho^2 - 1 + u)}}.$$

The semi-axes of the ellipsoid are

$$A^2 = \rho_0^2 - 1, \; B^2 = \rho_0^2 - \eta^2, \; C^2 = \rho_0^2.$$

The volume of the ellipsoid is

$$T = \frac{3}{4} \pi ABC.$$

Finally, we have

$$V = \pi ABC \int_{\lambda}^{\infty} \left(1 - \frac{x^2}{A^2 + u} - \frac{y^2}{B^2 + u} - \frac{z^2}{C^2 + u}\right) \frac{du}{\sqrt{(A^2 + u)(B^2 + u)(C^2 + u)}}.$$

Here the attraction constant is put equal to unity.

We get the formula for the attraction potential (on an exterior point) of a homogeneous ellipsoid. Dirichlet wrote it (21) in the form:

$$V = \pi \int_{\lambda}^{\infty} \left(1 - \frac{x^2}{A^2 + u} - \frac{y^2}{B^2 + u} - \frac{z^2}{C^2 + u}\right) \frac{du}{D}, \tag{52.1}$$

$$D = \sqrt{\left(1 + \frac{u}{A^2}\right)\left(1 + \frac{u}{B^2}\right)\left(1 + \frac{u}{C^2}\right)}.$$

We can consider the formula (44) as the expression of the attraction potential for a homogeneous triaxial ellipsoid.

CHAPTER II

1. THE GRAVITY ON THE SURFACE OF A LEVEL TRIAXIAL ELLIPSOID; GENERAL FORMULA

Let us obtain a formula for the gravity. Denoting by γ, the gravity on the surface of the ellipsoid $\rho = \rho_0$, we have

$$\gamma = -\frac{\partial U}{\partial n} = \left[-\frac{\partial U}{\partial \rho}\frac{d\rho}{dn}\right]_{\rho = \rho_0} \tag{53}$$

In accordance with the formula (8),

$$\frac{d\rho}{dn} = \iota\sqrt{(\rho^2 - 1)(\rho^2 - \eta^2)} = \iota R_1 R_2 .$$

Differentiating the fundamental formula for the potential of gravity (27) and taking into consideration the formula (14), we have

$$\frac{\partial S_7}{\partial \rho} = -\frac{5}{R_1 R_2 R_7} + \frac{S_7}{R_7} 2\rho,$$

$$\frac{\partial S_8}{\partial \rho} = -\frac{5}{R_1 R_2 R_8} + \frac{S_8}{R_8} 2\rho,$$

$$\frac{\partial S_0}{\partial \rho} = -\frac{1}{R_1 R_2},$$

$$\frac{\partial R_7}{\partial \rho} = 2\rho,$$

$$\frac{\partial R_8}{\partial \rho} = 2\rho,$$

The differentiation with respect to ρ yields

$$\left(\frac{\partial U}{\partial \rho}\right)_{\rho = \rho_0} = \left\{-\frac{M}{R_1^0 R_2^0} + \frac{2}{3}\omega^2 \rho^0\right\}$$
$$+ \left\{A_7\left(-\frac{5}{R_1^0 R_2^0 R_7^0} + \frac{S_7^0}{R_7^0} 2\rho_0\right) + \frac{\omega^2}{6(\alpha_2 - \alpha_1)}\left[\frac{2\rho_0}{1 - \alpha_1}\right]\right\} M_7 N_7 \tag{54}$$
$$+ \left\{A_8\left(-\frac{5}{R_1^0 R_2^0 R_8^0} + \frac{S_8^0}{R_8^0} 2\rho_0\right) + \frac{\omega^2}{6(\alpha_1 - \alpha_2)}\left[\frac{2\rho_0}{1 - \alpha_2}\right]\right\} M_8 N_8 .$$

In order to eliminate ω^2, we shall utilize the equations (30) of equilibrium. The first condition yields:

$$\frac{2}{3}\omega^2 = 4(\alpha_1 - \alpha_2) A_7 \frac{S_7^0}{\dfrac{R_7^0}{1-\alpha_1} + 2}$$

and

$$\frac{\omega^2}{6(\alpha_2 - \alpha_1)} = -\frac{A_7 S_7^0}{\dfrac{R_7^0}{1-\alpha_1} + 2} .$$

The second condition yields

$$\frac{\omega^2}{6(\alpha_1 - \alpha_2)} = -\frac{A_8 S_8^0}{\dfrac{R_8^0}{1-\alpha_2} + 2} .$$

The last expression obtained from the second condition of equilibrium can be put into the third brace.

Putting the three above-written expressions into formula (54) we obtain

$$\left(\frac{\partial U}{\partial \rho}\right)_{\rho=\rho_0} = -\frac{M}{R_1^0 R_2^0} + 4(\alpha_1 - \alpha_2)\rho_0 \frac{A_7 S_7^0}{\dfrac{R_7^0}{1-\alpha_1} + 2}$$

$$+ A_7\left\{-\frac{5}{R_1^0 R_2^0 R_7^0} + \frac{S_7^0}{R_7^0} 2\rho_0 - \left[\frac{2\rho_0}{1-\alpha_1}\right] \frac{S_7^0}{\dfrac{R_7^0}{1-\alpha_1} + 2}\right\} M_7 N_7$$

$$+ A_8\left\{-\frac{5}{R_1^0 R_2^0 R_8^0} + \frac{S_8^0}{R_8^0} 2\rho_0 - \left[\frac{2\rho_0}{1-\alpha_2}\right] \frac{S_8^0}{\dfrac{R_8^0}{1-\alpha_2} + 2}\right\} M_8 N_8 .$$

Multiplying by $\left(\dfrac{d\rho}{dn}\right)_{\rho=\rho_0} = \ell_0 R_1^0 R_2^0$ and changing sign and further taking into account that $\rho_0 = R_3^0$, and putting, following Lyapunov, $R_1^0 R_2^0 R_3^0 = \Delta$, we obtain:

$$\gamma = Ml - 4(\alpha_1 - \alpha_2)\, A_7 \Delta l_0 \frac{S_7^{\,0}}{\dfrac{R_7^{\,0}}{1-\alpha_1} + 2}$$

$$+ A_7 \left\{ \frac{5}{R_7^{\,0}} - 2\Delta \frac{S_7^{\,0}}{R_7^{\,0}} + \frac{2\Delta}{1-\alpha_1} \frac{S_7^{\,0}}{\dfrac{R_7^{\,0}}{1-\alpha_1} + 2} \right\} l_0 M_7 N_7 \tag{55}$$

$$+ A_8 \left\{ \frac{5}{R_8^{\,0}} - 2\Delta \frac{S_8^{\,0}}{R_8^{\,0}} + \frac{2\Delta}{1-\alpha_2} \frac{S_8^{\,0}}{\dfrac{R_8^{\,0}}{1-\alpha_2} + 2} \right\} l_0 M_8 N_8 .$$

Let us transform the first two terms in the first brace:

$$- 2\Delta \frac{S_7^{\,0}}{R_7^{\,0}} + \frac{2\Delta}{1-\alpha_1} \frac{S_7^{\,0}}{\dfrac{R_7^{\,0}}{1-\alpha_1} + 2} = - 2\Delta S_7^{\,0} \left[\frac{1}{R_7^{\,0}} - \frac{1}{1-\alpha_1} \frac{1}{\dfrac{R_7^{\,0}}{1-\alpha_1} + 2} \right]$$

$$= - 2\Delta S_7^{\,0} \left[\frac{2}{R_7^{\,0} \left(\dfrac{R_7^{\,0}}{1-\alpha_1} + 2 \right)} \right] = - 4 \frac{\Delta}{R_7^{\,0}} \left[\frac{S_7^{\,0}}{\dfrac{R_7^{\,0}}{1-\alpha_1} + 2} \right].$$

In analogous fashion the reduction of other terms in braces can be made.

Taking this condition into consideration and omitting the index "0", we finally have

$$\gamma = Ml - 4(\alpha_1 - \alpha_2)\, A_7 \left[\frac{S_7}{\dfrac{R_7}{1-\alpha_1} + 2} \right] \Delta l$$

$$+ \frac{A_7}{R_7} \left\{ \frac{5}{\Delta} - 4 \left[\frac{S_7}{\dfrac{R_7}{1-\alpha_1} + 2} \right] \right\} \Delta l M_7 N_7 \tag{56}$$

$$+ \frac{A_8}{R_8} \left\{ \frac{5}{\Delta} - 4 \left[\frac{S_8}{\dfrac{R_8}{1-\alpha_2} + 2} \right] \right\} \Delta l M_8 N_8 .$$

This is the general formula of gravity on a heterogeneous, triaxial, level ellipsoid expressed in Lame functions. The coefficients A_7 and A_8 must be determined by the equations (30).

2. TRANSITION TO A HOMOGENEOUS LEVEL ELLIPSOID; FORMULA OF POINCARÉ.

Now we shall prove that in a partial case if the planet in question is a homogeneous body, the gravity expressed in (56) can be reduced to the formula given by Poincare in the year 1885 [2]:

$$\gamma l = \frac{4}{3}\pi R_1 S_1. \tag{57}$$

We must remark that the magnitude l introduced by Liouville can be expressed in the following form:

$$\frac{1}{l^2} = (\rho^2 - \mu^2)(\rho^2 - \nu^2) = (R_7 - M_7)(R_7 - N_7) = (R_8 - M_8)(R_8 - N_8),$$

Hence

$$\frac{1}{l^2} = R_7^{\,2} - R_7\ (M_7 - N_7) + M_7N_7,$$

$$\frac{1}{l^2} = R_8^{\,2} - R_8\ (M_8 - N_8) + M_8N_8.$$

Multiplying the first equality by R_8 and the second by $-R_7$ and adding the results we obtain

$$\frac{1}{l^2}(R_8 - R_7) = R_7R_8\ (R_7 - R_8) + R_7R_8\ [(M_8 - M_7) + (N_8 - N_7)]$$
$$+ R_8M_7N_7 - R_7M_8N_8.$$

The magnitudes in parentheses are expressible in terms of the difference of the roots, therefore

$$\frac{1}{l^2}(\alpha_1 - \alpha_2) = R_7R_8(\alpha_2 - \alpha_1) + 2R_7R_8(\alpha_1 - \alpha_2) + R_8M_7N_7 - R_7M_8N_8.$$

Dividing this expression by the value $\alpha_1 - \alpha_2$, which is not equal to zero, we obtain

$$\frac{1}{l^2} = R_7R_8 + \frac{R_8}{\alpha_1 - \alpha_2}M_7N_7 - \frac{R_7}{\alpha_1 - \alpha_2}M_8N_8.$$

Thus we have the desired relation:

$$\frac{1}{\ell^2 R_7 R_8} = 1 + \frac{M_7 N_7}{R_7(\alpha_1 - \alpha_2)} + \frac{M_8 N_8}{R_8(\alpha_2 - \alpha_1)} . \tag{58}$$

Taking into consideration the equality (58) and the conditions of equilibrium, we can change the form of the general gravity formula:

$$\gamma = M\ell + 5\left(A_7 \frac{M_7 N_7}{R_7} + A_8 \frac{M_8 N_8}{R_8}\right)\ell - 4(\alpha_1 - \alpha_2)\, A_7 \Delta \frac{\dfrac{S_7}{\dfrac{R_7}{1 - \alpha_1} + 2}}{\ell R_7 R_8} . \tag{56.1}$$

In the case of a homogeneous body,

$$A_7 = \frac{T}{5(\alpha_1 - \alpha_2)} \text{ and } A_8 = \frac{T}{5(\alpha_2 - \alpha_1)} ,$$

Then (56) becomes

$$\gamma = M\ell - \frac{4}{5} T \left[\frac{S_7}{\dfrac{R_7}{1 - \alpha_1} + 2}\right] \Delta\ell + \frac{T}{5(\alpha_1 - \alpha_2)R_7}\left\{\frac{5}{\Delta} - 4\left[\frac{S_7}{\dfrac{R_7}{1 - \alpha_1} + 2}\right]\right\} \Delta\ell M_7 N_7$$

$$+ \frac{T}{5(\alpha_2 - \alpha_1)R_8}\left\{\frac{5}{\Delta} - 4\,\frac{S_8}{\dfrac{R_8}{1 - \alpha_2} + 2}\right\} \Delta\ell M_8 N_8 .$$

In the case of a homogeneous body the equilibrium conditions require the relation

$$\frac{S_7}{\dfrac{R_7}{1 - \alpha_1} + 2} = \frac{S_8}{\dfrac{R_8}{1 - \alpha_2} + 2} .$$

Consequently,